

Anti-self-dual four-manifolds with a parallel real spinor.

Maciej Dunajski*

The Mathematical Institute, 24-29 St Giles, Oxford OX1 3LB, UK

Abstract

Anti-self-dual metrics in the $(+ + --)$ signature which admit a covariantly constant real spinor are studied. It is shown that finding such metrics reduces to solving a fourth order integrable PDE, and some examples are given. The corresponding twistor space is characterised by existence of a preferred non-zero real section of $\kappa^{-1/4}$, where κ is the canonical line bundle of the twistor space. It is demonstrated that if the parallel spinor is preserved by a Killing vector, then the fourth order PDE reduces to the dispersionless Kadomtsev–Petviashvili equation and its linearisation. Einstein–Weyl structures on the space of trajectories of the symmetry are characterised by the existence of a parallel weighted null vector.

1 Introduction

Constraints on a (pseudo) Riemannian geometry imposed by the existence of a parallel spinor essentially depend on the properties of the Clifford algebra and the spin group associated with the metric. There has been an interest in such geometries in pure mathematics, because they extend a list of holonomy groups of Riemannian manifolds (where the existence of a parallel spinor implies Ricci flatness). Pseudo Riemannian metrics in various dimensions with a covariantly constant spinor have also attracted a lot of attention in physics, as such spinors play a central role in supersymmetry.

In [2] Bryant analysed all cases up to six dimensions, together with some higher-dimensional examples of particular interest. In this paper I shall concentrate on the four-dimensional case.

Let (\mathcal{M}, g) be a (pseudo) Riemannian spin four-manifold. Therefore there exist complex two-dimensional vector bundles S_{\pm} (spin-bundles) equipped with parallel symplectic structures ε_{\pm} such that

- $\mathbb{C} \otimes T\mathcal{M} \cong S_+ \otimes S_-$ is a canonical bundle isomorphism.
- $g(v_1 \otimes w_1, v_2 \otimes w_2) = \varepsilon_+(v_1, v_2)\varepsilon_-(w_1, w_2)$ for $v_1, v_2 \in \Gamma(S_+)$ and $w_1, w_2 \in \Gamma(S_-)$.

I shall assume that there exists a spinor ι parallel with respect to a Levi-Civita connection ∇ of g

$$\iota = (p, q) \in \Gamma(S_+), \quad \nabla \iota = 0.$$

There are three possible situations depending on the signature of the metric.

*email: dunajski@maths.ox.ac.uk

- In the Lorentzian signature $(+ - - -)$

$$\text{Spin}(3, 1) = SL(2, \mathbb{C}), \quad \iota = (p, q) \rightarrow \bar{\iota} = (\bar{p}, \bar{q}) \in \Gamma(S_-), \quad \nabla \bar{\iota} = 0.$$

Therefore $l = \iota \otimes \bar{\iota}$ is a parallel null vector. This condition has been extensively studied in general relativity [11]: There exist two real functions u, v , and one complex function ξ such that

$$g = du dv - d\xi d\bar{\xi} + H(u, \xi, \bar{\xi}) du^2.$$

The Ricci flat condition implies that $H(u, \xi, \bar{\xi}) = \text{Re}(f(u, \xi))$ where f is holomorphic in ξ . These solutions are known as *p.p waves*. Analysis of curvature invariants shows that the function $H(u, \xi, \bar{\xi})$ cannot be eliminated by a coordinate transformation.

- In the Euclidean signature $(+ + + +)$

$$\text{Spin}(4, 0) = SU(2) \times \widetilde{SU}(2), \quad \iota = (p, q) \rightarrow \bar{\iota} = (\bar{q}, -\bar{p}) \in \Gamma(S_+), \quad \nabla \bar{\iota} = 0.$$

A spinor and its complex conjugate form a basis of a spin space S_+ . A four-dimensional Riemannian manifold which admits a covariantly constant spinor must therefore be hyper-Kähler. Hyper Kähler four-manifolds have been much studied for the last twenty five years. See [3] and references therein.

- In the split signature $(+ + - -)$ (also called ultra-hyperbolic, Kleinian or neutral)

$$\text{Spin}(2, 2) = SL(2, \mathbb{R}) \times \widetilde{SL}(2, \mathbb{R}), \quad \iota = (p, q) \rightarrow \bar{\iota} = (\bar{p}, \bar{q}) \in \Gamma(S_+), \quad \nabla \bar{\iota} = 0,$$

and the representation space of the spin group splits into a direct sum of two real two-dimensional spin spaces S_+ and S_- . The conjugation of spinors is involutive and maps each spin space onto itself, and there exists an invariant notion of *real spinors*.

One can therefore look for $(+ + - -)$ metrics with a parallel real spinor (which we choose to be $\iota \in \Gamma(S_+)$). These metrics do not have to be Ricci-flat. The resulting geometry will be studied in the rest of this paper. The isomorphism $\Lambda^2_+(\mathcal{M}) \cong \text{Sym}^2(S_+)$ between the bundle of self-dual two-forms and the symmetric tensor product of two spin bundles implies that the real self-dual two-form $\Sigma = \iota \otimes \iota \otimes \varepsilon_+$ is covariantly constant and null (i.e. $\Sigma \wedge \Sigma = 0$) which motivates the following definition:

Definition 1.1 *A null-Kähler structure on a four-manifold consists of metric of signature $(+ + - -)$ and a real spinor field parallel with respect to this inner product. A null-Kähler structure is anti-self-dual (ASD) if the self-dual part of the Weyl spinor vanishes.*

The ASD condition on null-Kähler structures is worth studying for at least two reasons: firstly, four-dimensional vacuum metrics in signature $(+ + - -)$ appeared in describing the bosonic sector of the $N = 2$ super-string [14], [1]. Secondly, many integrable systems in dimensions two and three arise (together with their twistor description) as symmetry reductions of anti-self-duality equations on $(+ + - -)$ background [20, 13].

The first aspect will be not discussed in the present paper, but I shall reveal some connections with integrable systems in Sections 3 and 5.

In the next Section the ASD null Kähler condition will be related to Einstein-Maxwell equations. In Section 3 ASD null-Kähler condition will be reduced to a single fourth order integrable PDE. Explicit solutions to this equations will provide some examples of ASD

null-Kähler structures. The resulting twistor theory will be described in Section 4. The existence of a parallel real spinor will be characterised by a real structure preserving a preferred non-zero section of $\kappa^{-1/4}$, where κ is the canonical line bundle of the twistor space. In Section 5 it will be shown that ASD null Kähler structures with a symmetry which preserves the parallel spinor are locally given by solutions to the dispersionless Kadomtsev–Petviashvili equation and its linearisations [4]. Einstein–Weyl structures on the space of trajectories of the symmetry will be characterised by the existence of a parallel weighted null vector. The two-component spinor notation will be used in the paper. The spin spaces S_- , and S_+ will be denoted by S^A and $S^{A'}$ respectively. From now on the parallel real spinor and the corresponding null-Kähler two-form will be denoted by $\iota^{A'} \in \Gamma(S^{A'})$ and $\Sigma^{0'0'} = \iota_{A'} \iota_{B'} \Sigma^{A'B'} \in \Lambda^2_+(\mathcal{M})$. The notation is summarised in the Appendix.

2 Null-Kähler metrics in four dimensions

It is well known [18] that Kähler four-manifolds with vanishing scalar curvature are necessary ASD. This is not true for scalar-flat null-Kähler four-manifolds. Instead one has the following result:

Proposition 2.1 *Let $\iota_{A'}$ be a parallel real spinor on dimensional, ultrahyperbolic manifold. Then the scalar curvature vanishes, the Ricci tensor is null, and the self-dual Weyl spinor is given by*

$$C_{A'B'C'D'} = c \iota_{A'} \iota_{B'} \iota_{C'} \iota_{D'} \quad (2.1)$$

for some function c such that $\iota^{A'} \nabla_{AA'} c = 0$.

Proof: Vanishing of the scalar curvature follows from the second Ricci identity (7.39). The first identity (7.38) implies that $\Phi_{ABA'B'} \iota^{B'} = 0$, so $\Phi_{ABA'B'} = F_{AB} \iota_{A'} \iota_{B'}$ for some F_{AB} . The formula 2.1 is a direct consequence of (7.39) and (7.40) applied to a covariantly constant spinor. □

Now I shall show that ASD null-Kähler metrics can be viewed as solutions to Einstein equations with electromagnetic stress-energy tensor. The ASD part of the Maxwell field is given by the Ricci form, and the SD part is given by the null-Kähler form.

Proposition 2.2 *There is a one-to-one correspondence between ASD metrics with a constant real spinor, and ASD Einstein-Maxwell spaces for which the SD part of Maxwell field is null and covariantly constant.*

Proof: Let (\mathcal{M}, g) be an ASD manifold with a covariantly constant spinor $\iota_{A'}$. Proposition 2.1 implies that

$$\Phi_{ABA'B'} = F_{AB} \iota_{A'} \iota_{B'}, \quad (2.2)$$

and the spinor Bianchi identities (7.40) yield $\nabla^{AA'} F_{AB} = 0$. Therefore

$$F_{ab} = F_{AB} \varepsilon_{A'B'} + \iota_{A'} \iota_{B'} \varepsilon_{AB}. \quad (2.3)$$

is a Maxwell field, and the formula (2.2) can be read off as Einstein equations with a Maxwell stress energy tensor

$$T_{ab} = \frac{1}{2} \left(\frac{1}{4} g_{ab} F_{cd} F^{cd} - F_{ac} F_b{}^c \right) = F_{AB} \iota_{A'} \iota_{B'}.$$

Conversely, consider a Maxwell field F_{ab} (2.3) on an ASD background, such that its SD part is null and constant. The Maxwell equations give

$$\nabla_{A'}^A F_{AB} = 0, \quad \nabla_{AA'} \iota_{B'} = 0,$$

and the Einstein equations with the stress energy tensor $T_{ab} = F_{AB} \iota_{A'} \iota_{B'}$ yield (2.2). \square

The following result will be used in Sections 3 and 4

Proposition 2.3 *Let $\Sigma^{A'B'} = (\Sigma^{0'0'}, \Sigma^{0'1'}, \Sigma^{1'1'})$ be a basis of normalised real SD two-forms on an ASD scalar-flat manifold such that*

$$d(\iota_{A'} \iota_{B'} \Sigma^{A'B'}) = d(o_{A'} \iota_{B'} \Sigma^{A'B'}) = 0. \quad (2.4)$$

Then there exists a covariantly constant real section of $S_{A'}$. Conversely, let $\iota_{A'}$ be a covariantly constant section of $S_{A'}$ on an ASD four manifold. Then it is possible to find another section $o_{A'}$, such that $(o_{A'}, \iota_{A'})$ form a normalised spin frame, and equations (2.4) hold.

Proof: Let $(o_{A'}, \iota_{A'})$ be a normalised spin basis. The covariant derivatives of the basis can be expressed as

$$\nabla_a \iota_{B'} = U_a \iota_{B'} + V_a o_{B'}, \quad \nabla_a o_{B'} = W_a \iota_{B'} - U_a o_{B'}.$$

The first condition in (2.4) can be rewritten as $\nabla_A^{A'} (\iota_{A'} \iota_{B'}) = 0$, which implies

$$V_a = 2U_{AB'} \iota^{B'} \iota_{A'}.$$

The second condition in (2.4) yields $U_{AA'} = \alpha_A \iota_{A'}, W_{AA'} = \beta_A \iota_{A'}$ for some α_A, β_A . Therefore

$$\nabla_{AA'} \iota_{B'} = \alpha_A \iota_{A'} \iota_{B'}. \quad (2.5)$$

Contracting the RHS of the above equation with $\nabla^A_{C'}$, and symmetrising over $(A'B')$ gives 0 because g is ASD and scalar-flat. As a consequence $\nabla^A_{(C'} [\iota_{A')} \iota_{B'} \alpha_A] = 0$ which gives $\nabla^A_{A'} \alpha_A = 0$. Consider the real spinor $\hat{\iota}_{A'} := \iota_{A'} \exp f$, where $\iota^{A'} \nabla_{AA'} f = 0$ in order to preserve $d\Sigma^{0'0'} = 0$. Integrability conditions for $\nabla_{AA'} f = \alpha_A \iota_{A'}$ are satisfied, as α_A solves the neutrino equation. Therefore we can find f for each α_A , and equation (2.5) implies that $\hat{\iota}_{A'}$ is covariantly constant.

Converse. Now assume $\nabla_{AA'} \iota_{B'} = 0$. Consider $\hat{o}_{A'}$ such that $\hat{o}_{A'} \iota^{A'} = 1$. The normalization condition implies $\nabla_{AA'} \hat{o}_{B'} = \gamma_A \iota_{A'} \iota_{B'} + \rho_A \hat{o}_{A'} \iota_{B'}$ for some γ_A, ρ_A . Contracting with $\nabla^A_{C'}$ and using Ricci identities (7.38) yields $\nabla^A_{A'} \rho_A = 0$, and so $\rho_A = \iota^{A'} \nabla_{AA'} \phi$. Therefore in the null rotated normalised spin frame $\iota_{A'}, \hat{o}_{A'} - \phi \iota_{A'}$ we have $\rho_A = 0$. Therefore

$$\nabla_A^{A'} (o_{A'} \iota_{B'} + o_{B'} \iota_{A'}) = 2W_A^{A'} \iota_{A'} \iota_{B'} = \rho_A \iota_{B'} = 0,$$

and $d\Sigma^{0'1'} = 0$. \square

3 Anti-Self-Dual null-Kähler condition as an integrable system

I shall now construct a local coordinate system adapted to the parallel spinor, and reduce the ASD null-Kähler condition to a pair of coupled PDEs. Integrability of these PDEs will be established by using the Lax formulation (i.e. showing that they arise as the integrability conditions to an over-determined system of linear equations).

Let $S^{A'} = \mathcal{M} \times \mathbb{C}^2$ be the bundle of complex primed spinors. The natural context for introducing the Lax pair is the geometry of the projective primed spin bundle (also called the correspondence space) $\mathcal{F} = \mathbb{P}(S^{A'}) = \mathcal{M} \times \mathbb{CP}^1$. It is coordinatized by (x^a, λ) , where x^a denotes the coordinates on \mathcal{M} and λ is the coordinate on \mathbb{CP}^1 that parametrises the α -surfaces through x in \mathcal{M} . We relate the fibre coordinates $\pi^{A'}$ on $S^{A'}$ to λ by $\lambda = \pi_{0'}/\pi_{1'}$.

Let $\nabla_{AA'}$ be a null tetrad of vector fields for the metric g on \mathcal{M} and let $\Gamma_{AA'B'C'}$ be the components of the spin connection in the associated spin frame. A horizontal lift of $\nabla_{AA'}$ to $S^{A'}$ given by

$$\tilde{\nabla}_{AA'} = \nabla_{AA'} + \Gamma_{AA'B'C'} \pi^{B'} \frac{\partial}{\partial \pi^{C'}}$$

Its horizontality implies $\tilde{\nabla}_{AA'} \pi^{B'} = 0$.

The space \mathcal{F} possesses a natural two-dimensional distribution called the twistor distribution, or Lax pair to emphasise the analogy with integrable systems. The Lax pair arises as the image under the projection $TS^{A'} \rightarrow T\mathcal{F}$ of the distribution spanned by $L_A = \pi^{A'} \tilde{\nabla}_{AA'}$, and is given by

$$L_0 = \nabla_{00'} - \lambda \nabla_{01'} + l_0 \partial_\lambda, \quad L_1 = \nabla_{10'} - \lambda \nabla_{11'} + l_1 \partial_\lambda, \quad (3.6)$$

where $l_A = \Gamma_{AA'B'C'} \pi^{A'} \pi^{B'} \pi^{C'}$ are cubic polynomials in λ (note that $\pi_{1'} = 1$ in these formulae).

Theorem 3.1 [15] *The twistor distribution on $S^{A'}$ given by (3.6) is integrable if and only if the Weyl curvature of g is ASD, i.e. $C_{A'B'C'D'} = 0$.*

We are now ready to reformulate the ASD null-Kähler condition as an integrable system.

Theorem 3.2 *Real coordinates (w, z, x, y) can be chosen such that all ASD null-Kähler metric are locally given by*

$$g = dw dx + dz dy - \Theta_{xx} dz^2 - \Theta_{yy} dw^2 + 2\Theta_{xy} dw dz \quad (3.7)$$

where $\Theta(w, z, x, y)$ is a solution to a 4th order PDE (which we write as a system of two second order PDEs):

$$\Theta_{wx} + \Theta_{zy} + \Theta_{xx} \Theta_{yy} - \Theta_{xy}^2 = f, \quad (3.8)$$

$$\square f = f_{xw} + f_{yz} + \Theta_{yy} f_{xx} + \Theta_{xx} f_{yy} - 2\Theta_{xy} f_{xy} = 0. \quad (3.9)$$

Moreover (3.8,3.9) arise as an integrability condition for the linear system $L_0 \Psi = L_1 \Psi = 0$, where $\Psi = \Psi(w, z, x, y, \lambda)$ and

$$\begin{aligned} L_0 &= (\partial_w - \Theta_{xy} \partial_y + \Theta_{yy} \partial_x) - \lambda \partial_y + f_y \partial_\lambda, \\ L_1 &= (\partial_z + \Theta_{xx} \partial_y - \Theta_{xy} \partial_x) + \lambda \partial_x - f_x \partial_\lambda. \end{aligned} \quad (3.10)$$

Proof: Let $e^{AA'}$ be a tetrad of real independent one-forms. The parallel spinor $\iota_{A'}$ enables us to choose coordinates $w^A = (w, z)$ such that $e^{A0'} = -\iota_{A'}e^{AA'} = dw^A$. Proposition 2.3 implies that we can choose $o_{A'}$ such that $o_{A'}\iota^{A'} = 1$, and

$$\Sigma^{0'1'} = \frac{1}{2}\varepsilon_{AB}o_{A'}\iota_{B'}e^{AA'} \wedge e^{BB'} = o_{A'}e^{AA'} \wedge dw_A$$

is a closed two-form. Therefore the Frobenius theorem guarantees the existence of coordinates $x_A = (x, y)$ such that $e_A^{1'} = o_{A'}e_A^{A'} = dx_A + \Theta_{AB}dw^B$, where $\Theta_{AB} = \Theta_{AB}(w, z, x, y)$ is symmetric in A and B . With this choice $\Sigma^{0'1'} = dx_A \wedge dw^A$, and the metric is given by

$$g = dx_A dw^A + \Theta_{AB}dw^A dw^B.$$

Calculating the components of the spin connection yields $\Gamma_{AA'B'C'} = A_{AA'}\iota_{B'}\iota_{C'}$. The residual conformal freedom is used to set $A_{AA'} = \beta_A\iota_{A'}$.

The tetrad of vector fields $\nabla_{AA'}$ dual to $e^{AA'}$ is

$$\nabla_{A1'} = \iota^{A'}\nabla_{AA'} = \frac{\partial}{\partial x^A}, \quad \nabla_{A0'} = o^{A'}\nabla_{AA'} = \frac{\partial}{\partial w^A} + \Theta_{AB}\frac{\partial}{\partial x_B},$$

and the Lax pair (3.6) is

$$L_A = \frac{\partial}{\partial x^A} - \lambda\left(\frac{\partial}{\partial w^A} + \Theta_{AB}\frac{\partial}{\partial x_B}\right) + l_A\frac{\partial}{\partial \lambda},$$

where in the chosen spin frame $l_A = \beta_A$ do not depend on λ . Consider the Lie bracket

$$[L_0, L_1] = \left(\frac{\partial\Theta^{CD}}{\partial w^C} + \Theta_{AC}\frac{\partial\Theta^{AD}}{\partial x_C} - \beta^D - \lambda\frac{\partial\Theta^{AD}}{\partial x^A}\right)\frac{\partial}{\partial x^D} + \left(\frac{\partial\beta^A}{\partial w^A} + \Theta_{AB}\frac{\partial\beta^A}{\partial x_B} + \lambda\frac{\partial\beta_A}{\partial x_A}\right)\frac{\partial}{\partial \lambda}.$$

The ASD condition is equivalent to integrability of the distribution L_A . In fact $[L_0, L_1] = 0$ because there is no $\partial/\partial w^A$ term in the Lie bracket above. We deduce that

$$\Theta_{AB} = \delta_A\delta_B\Theta, \quad \beta_A = \delta_A f, \quad \text{where } \delta_A := \iota^{A'}\nabla_{AA'} = \frac{\partial}{\partial x^A},$$

and $f = f(w, z, x, y)$ and $\Theta(w, z, x, y)$ satisfy

$$\Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = f + \mathcal{F}(w, z), \quad \square f = 0.$$

To obtain (3.8) we absorb $\mathcal{F}(w, z)$ into f without changing (3.9).

□

Remarks

- One-forms $(e^{00'}, e^{10'})$ span a differential ideal, and $\Sigma^{0'0'} = e^{00'} \wedge e^{10'} = dw \wedge dz$ is the null-Kähler form (it is covariantly constant with respect to the metric (3.7)). On the other hand $(e^{01'}, e^{11'})$ do not span an ideal unless $f = 0$ in which case g is pseudo-hyper-Kähler.
- The components of the Ricci and Weyl curvatures, and the Levi-Civita spinor connection are

$$\begin{aligned} C_{ABCD} &= \delta_A\delta_B\delta_C\delta_D\Theta, & C_{A'B'C'D'} &= \iota_{A'}\iota_{B'}\iota_{C'}\iota_{D'}\square f = 0, & R &= 0, \\ \Phi_{ABA'B'} &= \iota_{A'}\iota_{B'}\delta_A\delta_B f, & \Gamma_{AB} &= \delta_A\delta_B\delta_C\Theta dw^C, & \Gamma_{A'B'} &= \iota_{A'}\iota_{B'}\delta_A f dw^A. \end{aligned}$$

The Bianchi identity (7.40) is satisfied as a consequence of equation (3.9). To sum out f is a potential for a null Maxwell field (so called Hertz potential), and Θ is a non-linear potential for a metric.

Potential forms of complexified null Einstein-Maxwell equations were given in [7] and [19]. It will be instructive to look for a non-trivial overlap between them and the one given above.

3.1 Examples

(1) Consider a class of metrics given by $\Theta_x = 0$. Equations (3.8,3.9) reduce to

$$\Theta_{yz} = f, \quad f_{yz} = 0.$$

The general solution is given by

$$\Theta = B(w, y) + z \int A(w, y) dy, \quad f = A(w, y), \quad (3.11)$$

where $A(w, y)$ and $B(w, y)$ are arbitrary functions. (In fact there are other terms linear in y and depending on arbitrary functions of w, z . These terms can be gauged away, as they do not change the metric.) We have

$$g = dw dx + dz dy - (z A_y + B_{yy}) dw^2.$$

If $A_y = 0$ then g is pseudo hyper-Kähler.

(2) Consider solutions with $f = \Theta_v$, where v is one of (w, z, x, y) . Equation (3.9) implies

$$\frac{\partial}{\partial v} (\Theta_{wx} + \Theta_{zy} + \Theta_{xx} \Theta_{yy} - \Theta_{xy}^2) = 0,$$

and differentiating equation (3.8) yields $\Theta_{vv} = 0$. It is enough to consider two sub-cases : $\Theta_{xx} = 0$ and $\Theta_{ww} = 0$ where the former one can be integrated explicitly

$$g = dw dx + dz dy - (x P_y + z(P - P_w + 2P P_y) + Q) dw^2 + 2P dw dz, \quad (3.12)$$

where $P(w, y)$ and $Q(w, y)$ are arbitrary functions.

(3)

$$\begin{aligned} \Theta &= A\left(\frac{x}{y}\right), \\ g &= dw dx + dz dy - \frac{1}{y^2} \left(\left(\frac{x^2}{y^2} A'' + 2\frac{x}{y} A'\right) dw^2 + A'' dz^2 + 2\left(\frac{x}{y} A'' + A'\right) dw dz \right) \end{aligned} \quad (3.13)$$

where A is an arbitrary function, and A' is its derivative.

(4)

$$\begin{aligned} \Theta &= x A(y) + B(y), \\ g &= dw dx + dz dy - (x A_{yy} + B_{yy}) dw^2 + 2A_y dw dz \end{aligned} \quad (3.14)$$

where A and B are arbitrary functions of one variable.

4 Twistor theory of ASD Null-Kähler metrics

All ASD null-Kähler metrics locally arise from solutions to (3.8,3.9). Non-analytic solutions are generic in (2,2) signature. However, in order to find a twistor description, in this section I shall restrict myself to real-analytics solutions.

Given an analytic solution to (3.8,3.9) one can obtain the corresponding twistor space by equipping $\mathcal{M} \times \mathbb{CP}^1$ with an integrable complex structure: The basis of $[0,1]$ vectors is $(L_0, L_1, \partial_{\bar{\lambda}})$, where (L_0, L_1) are given by (3.10). The parallel spinor $\iota^{A'}$ gives rise to the section $l = \iota^{A'} \pi_{A'}$ of $\kappa^{-1/4}$. In this section I shall perform this construction (together with its converse) in a coordinate independent way.

Definition 4.1 *An α -surface is a totally null two-dimensional surface, such that a two-form orthogonal to its tangent plane is SD.*

There are Frobenius integrability conditions for the existence of such α -surfaces through each point and these are equivalent, by Theorem 3.1 to the vanishing of the self-dual part of the Weyl curvature, $C_{A'B'C'D'}$. Thus, given $C_{A'B'C'D'} = 0$, we can define a twistor space \mathcal{PT} to be the three complex dimensional manifold of α -surfaces in \mathcal{M} .

A tangent space to an α -surface is spanned by null vectors of the form $\lambda^A \pi^{A'}$ with $\pi^{A'}$ fixed and λ^A arbitrary. As mentioned in the introduction, in the split signature any spinor has an invariant decomposition into its real and imaginary part. A *real α -surface* corresponds to both λ^A and $\pi^{A'}$ being real.

In general $\pi^{A'} = \text{Re}\pi^{A'} + i\text{Im}\pi^{A'}$, and the correspondence space \mathcal{F} defined in the last section decomposes into two open sets

$$\begin{aligned}\mathcal{F}_+ &= \{(x^a, [\pi^{A'}]) \in \mathcal{F}; \text{Re}(\pi_{A'})\text{Im}(\pi^{A'}) > 0\} = \mathcal{M} \times D_+, \\ \mathcal{F}_- &= \{(x^a, [\pi^{A'}]) \in \mathcal{F}; \text{Re}(\pi_{A'})\text{Im}(\pi^{A'}) < 0\} = \mathcal{M} \times D_-, \end{aligned}$$

where D_{\pm} are two copies of a Poincare disc. These complex submanifolds are separated by a real correspondence space

$$\mathcal{F}_0 = \{(x^a, [\pi^{A'}]) \in \mathcal{F}; \text{Re}(\pi_{A'})\text{Im}(\pi^{A'}) = 0\} = \mathcal{M} \times \mathbb{RP}^1.$$

The vector fields (3.6) together with the complex structure on the \mathbb{CP}^1 give \mathcal{F} a structure of a complex manifold \mathcal{PT} : The integrable sub-bundle of $T\mathcal{F}$ is spanned by $L_0, L_1, \partial_{\bar{\lambda}}$. The distribution (3.6) with $\lambda \in \mathbb{RP}^1$ define a foliation of \mathcal{F}_0 with a quotient \mathcal{PT}_0 which leads to a double fibration:

$$\mathcal{M} \xleftarrow{p} \mathcal{F}_0 \xrightarrow{q} \mathcal{PT}_0. \quad (4.15)$$

The twistor space \mathcal{PT} is a union of two open subsets $\mathcal{PT}_+ = (\mathcal{F}_+)$ and $\mathcal{PT}_- = (\mathcal{F}_-)$ separated by a three-dimensional real boundary¹ (*real twistor space*) $\mathcal{PT}_0 := q(\mathcal{F}_0)$.

The *real structure* $\sigma(x^a) = \bar{x}^a$ maps α -surfaces to α -surfaces, and therefore induces an anti-holomorphic involution $\sigma : \mathcal{PT} \rightarrow \mathcal{PT}$. The fixed points of this involution correspond to real α -surfaces in \mathcal{M} . There is an \mathbb{RP}^1 worth of such α -surfaces through each point of \mathcal{M} . The set of fixed points of σ in \mathcal{PT} is \mathcal{PT}_0 .

¹ In [22] Woodhouse performed a careful analysis of the twistor correspondence for flat (+ + --) metrics, and showed how functions on $\mathcal{PT}_0 = \mathbb{RP}^3$ can be used to construct smooth solutions to the ultra-hyperbolic wave equation.

Each point $x \in \mathcal{M}$ determines a sphere l_x made up of all the α -surfaces through x . The normal bundle of l_x in \mathcal{PT} is $N = T\mathcal{PT}|_{l_x}/Tl_x$. This is a rank-two vector bundle over \mathbb{CP}^1 , therefore it has to be one of the standard line² bundles $\mathcal{O}(n) \oplus \mathcal{O}(m)$.

Lemma 4.2 *Let $p : \mathcal{F} = \mathcal{M} \times \mathbb{CP}^1 \longrightarrow \mathcal{M}$. The holomorphic curves $l_x := p^{-1}(x)$, $x \in \mathcal{M}$ have normal bundle $N = \mathcal{O}(1) \oplus \mathcal{O}(1)$.*

Proof. The bundle N can be identified with the quotient $p^*(T_x\mathcal{M})/\{\text{span } L_0, L_1\}$. In their homogeneous form the operators L_A have weight one, so the distribution spanned by them is isomorphic to the bundle $\mathbb{C}^2 \otimes \mathcal{O}(-1)$. The definition of the normal bundle as a quotient gives a sequence of sheaves over \mathbb{CP}^1 .

$$0 \longrightarrow \mathbb{C}^2 \otimes \mathcal{O}(-1) \longrightarrow \mathbb{C}^4 \longrightarrow N \longrightarrow 0$$

and we see that $N = \mathcal{O}(1) \oplus \mathcal{O}(1)$, because the last map, in the spinor notation, is given explicitly by $V^{AA'} \mapsto V^{AA'} \pi_{A'}$ projecting onto $\mathcal{O}(1) \oplus \mathcal{O}(1)$. □

If \mathcal{M} is ASD null-Kähler then \mathcal{PT} has an additional structure:

Theorem 4.3 *Let \mathcal{PT} be a three-dimensional complex manifold with*

- *a four parameter family of rational curves with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$,*
- *a preferred section of $\kappa^{-1/4}$, where κ is the canonical bundle of \mathcal{PT} ,*
- *an anti-holomorphic involution $\rho : \mathcal{PT} \rightarrow \mathcal{PT}$ fixing a real equator of each rational curve, and leaving the section of $\kappa^{-1/4}$ above invariant,*

Then the real moduli space \mathcal{M} of the ρ -invariant curves is equipped with a restricted conformal class $[g]$ of ASD null-Kähler metric: If $g \in [g]$ and $\Sigma^{0'0'}$ is a null-Kähler two-form then $\hat{g} = \Omega^2 g \in [g]$ for any Ω such that $d\Omega \wedge \Sigma^{0'0'} = 0$. Conversely, given a real analytic ASD null Kähler metric, there exists a corresponding twistor space with the above structures.

Proof: Let g be a real analytic ASD metric with a covariantly constant real spinor $\iota_{A'}$. From $C_{A'B'C'D'} = 0$ it follows that there exist coordinates $\pi^{A'}$ on the fibres of $S^{A'} \rightarrow \mathcal{M}$ such that $\pi^{A'} \widetilde{\nabla}_{AA'} \pi^{B'} = 0$. Therefore a parallel section $\iota_{A'}$ of $S_{A'}$ determines a function $l = \pi^{A'} \iota_{A'}$ constant along the twistor distribution. The line bundle $\pi^{A'} \iota_{A'} = 0$ on \mathcal{PT} is isomorphic to $\kappa^{-1/4}$, where $\kappa = \Omega^3 \mathcal{PT}$ is the canonical bundle.

Converse. The global section l of $\kappa^{-1/4}$, when pulled back to $S_{A'}$ determines a homogeneity degree one function on each fibre of $S_{A'}$ and so must, by globality, be given by $l = \iota^{A'} \pi_{A'}$ and since l is pulled back from twistor space, it must satisfy $\pi^{A'} \widetilde{\nabla}_{AA'} l = 0$. This implies

$$\nabla_{AA'} \iota_{B'} = \varepsilon_{A'B'} \alpha_A \tag{4.16}$$

for some α_A . Choose a representative in $[g]$ with $R = 0$. Contracting (4.16) with $\nabla^A_{C'}$ and using the spinor Bianchi identity gives

$$\nabla^A_{C'} \nabla_{AA'} \iota_{B'} = C_{A'B'C'D'} \iota^{D'} - \frac{1}{12} R \varepsilon_{C'(B'} \iota_{A')} = 0 = \varepsilon_{A'B'} \nabla^A_{C'} \alpha_A,$$

² Here $\mathcal{O}(n)$ denotes the line bundle over \mathbb{CP}^1 with transition functions λ^{-n} from the set $\lambda \neq \infty$ to $\lambda \neq 0$ (i.e. Chern class n).

so α_A is a solution to the ASD spin 1/2 equation $\nabla^{AA'}\alpha_A = 0$ (so-called neutrino equation). It can be written in terms of a potential

$$\alpha_A = \iota^{A'} \nabla_{AA'} \phi \quad (4.17)$$

because the integrability conditions $\iota^{A'} \iota^{B'} \nabla^A_{A'} \alpha_A = \alpha_A \iota^{A'} \nabla^A_{A'} \iota^{B'}$ are satisfied. Here ϕ is a real analytic function which satisfies

$$\nabla^a \nabla_a \phi + \nabla_a \phi \nabla^a \phi = 0 \quad (4.18)$$

as a consequence of the neutrino equation. Consider a conformal rescaling

$$\hat{g} = \Omega^2 g, \quad \hat{\varepsilon}_{A'B'} = \Omega \varepsilon_{A'B'}, \quad \hat{\iota}_{A'} = \Omega \iota_{A'}, \quad \hat{\iota}^{A'} = \iota^{A'}, \quad \hat{R} = R + \frac{1}{4} \Omega^{-1} \square \Omega.$$

The twistor equation (4.16) is conformally invariant as $\hat{\nabla}_A^{(A'} \hat{\iota}^{B')} = \Omega^{-1} \nabla_A^{(A'} \iota^{B')} = 0$. Choose $\Omega \in \ker \square$ so that $\hat{R} = 0$. Let $\Upsilon_a = \Omega^{-1} \nabla_a \Omega$. Then

$$\hat{\nabla}_{AA'} \hat{\iota}^{B'} = \nabla_{AA'} \iota^{B'} + \varepsilon_{A'}^{B'} \Upsilon_{AB'} \iota^{C'} = \varepsilon_{A'}^{B'} (\iota^{C'} \nabla_{AC'} (\phi + \ln \Omega))$$

where we used (4.16) and (4.17). Notice that, as a consequence of (4.18), $\exp(\phi) \in \ker \square$ and we can choose $\ln \Omega = -\phi$, and

$$\hat{\nabla}_{AA'} \hat{\iota}^{B'} = 0. \quad (4.19)$$

We can still use the residual gauge freedom and add to ϕ and an arbitrary function Ω constant along $\iota^{A'} \nabla_{AA'}$, which by Frobenius theorem implies $d\Omega \wedge \Sigma^{0'0'} = 0$. This means (4.19) is invariant under a conformal rescaling by functions constant along the leaves of the congruence defined by $\hat{\iota}^{A'}$. Such conformal transformations do not change $\hat{R} = 0$.

□

5 ASD null Kähler metrics with symmetry

In this section I shall consider ASD null Kähler metrics which admit a Killing vector preserving the parallel spinor. Let us call them *ASD null Kähler metrics with symmetry*. I shall show that all such metrics are (at least in the real analytic case) locally determined by solutions to a certain integrable equation and its linearisation.

Before establishing this result I shall review some facts about Einstein–Weyl (EW) spaces which admit a parallel weighted vector[5].

5.1 3D Einstein–Weyl spaces with a parallel weighted vector

Let \mathcal{W} be a three-dimensional real manifold with a torsion-free connection D and a conformal metric $[h]$. We shall call \mathcal{W} a Weyl space if the null geodesics of $[h]$ are also geodesics for D . This condition is equivalent to

$$Dh = \nu \otimes h \quad (5.20)$$

for some one form ν . Here h is a representative metric in the conformal class. If we change this representative by $h \longrightarrow \phi^2 h$, then $\nu \longrightarrow \nu + 2d \ln \phi$. A tensor object T which transforms as $T \longrightarrow \phi^m T$ when $h \longrightarrow \phi^2 h$ is said to be conformally invariant of weight m . The covariant derivative of a one-form β of weight m can be expressed in terms of the Levi-Civita connection of h :

$$\tilde{D}\beta = \nabla\beta - \frac{1}{2}(\beta \otimes \nu + (1-m)\nu \otimes \beta - h(\nu, \beta)h). \quad (5.21)$$

The conformally invariant Einstein–Weyl (EW) condition on (\mathcal{W}, h, ν) is

$$W_{(ij)} = \frac{1}{3}\Lambda h_{ij}.$$

Where Λ and W_{ij} are the scalar curvature, and the Ricci tensor of the Weyl connection.

Three-dimensional EW structures are related to four-dimensional ASD conformal structures by the Jones-Tod correspondence [10]:

Proposition 5.1 ([10]) *Let (\mathcal{M}, \hat{g}) be a ultra-hyperbolic four-manifold with ASD conformal curvature, and a conformal Killing vector K . The EW structure in indefinite signature on the space \mathcal{W} of trajectories of K is defined by*

$$h := |K|^{-2}\hat{g} - |K|^{-4}\mathbf{K} \odot \mathbf{K}, \quad \nu := 2|K|^{-2} *_\hat{g}(\mathbf{K} \wedge d\mathbf{K}), \quad (5.22)$$

where $|K|^2 := \hat{g}_{ab}K^a K^b$, \mathbf{K} is the one form dual to K and $*_\hat{g}$ is taken with respect to \hat{g} . All three-dimensional EW structures arise in this way.

Conversely, let (h, ν) be a three-dimensional EW structure with a signature $(++-)$ on \mathcal{W} , and let (V, α) be a pair consisting of a function of weight -1 and a one-form on \mathcal{W} which satisfy the generalised monopole equation

$$*_h(dV + (1/2)\nu V) = d\alpha, \quad (5.23)$$

where $*_h$ is taken with respect to h . Then

$$g = Vh - V^{-1}(dz + \alpha)^2 \quad (5.24)$$

is an ASD metric with an isometry $K = \partial_z$.

In [5] it has been demonstrated that if an EW space admits a parallel weighted vector, the coordinates can be found in which the metric and the one form are given by

$$h = dy^2 - 4dxdt - 4udt^2, \quad \nu = -4u_x dt, \quad u = u(x, y, t) \quad (5.25)$$

and the EW equations reduce to the dispersionless Kadomtsev–Petviashvili equation

$$(u_t - uu_x)_x = u_{yy}. \quad (5.26)$$

If $u(x, y, t)$ is a smooth real function of real variables then (5.25) has signature $(++-)$. It has also been shown that there exists a twistor construction of EW spaces (5.25) given by the following Theorem

Theorem 5.2 [5] *There is a one to one correspondence between Einstein–Weyl spaces (5.25) obtained from solutions to the equation (5.26) and two-dimensional complex manifolds with*

- A three parameter family of rational curves with normal bundle $\mathcal{O}(2)$.
- A global section l of $\kappa^{-1/4}$, where κ is the canonical bundle.
- An anti-holomorphic involution fixing a real slice, leaving a rational curve and the preferred section of $\kappa^{-1/4}$ invariant.

5.2 Symmetry reduction

Now we are ready to establish the main result of this section.

Theorem 5.3 *Let $H = H(x, y, t)$ and $W = W(x, y, t)$ be smooth real-valued functions on an open set $\mathcal{W} \subset \mathbb{R}^3$ which satisfy³*

$$H_{yy} - H_{xt} + H_x H_{xx} = 0, \quad (5.27)$$

$$W_{yy} - W_{xt} + (H_x W_x)_x = 0. \quad (5.28)$$

Then

$$g = W_x(dy^2 - 4dxdt - 4H_x dt^2) - W_x^{-1}(dz - W_x dy - 2W_y dt)^2 \quad (5.29)$$

is an ASD null Kähler metric on a circle bundle $\mathcal{M} \rightarrow \mathcal{W}$. All real analytic ASD null Kähler metrics with symmetry arise from this construction.

Proof. Let (h, ν) be a three-dimensional EW structure given by (5.25) (with $u = H_x$) and let (V, α) be a pair consisting of a function and a one-form which satisfy the generalised monopole equation (5.23). The ultra-hyperbolic metric

$$g = V(dy^2 - 4dxdt - 4H_x dt^2) - V^{-1}(dz + \alpha)^2 \quad (5.30)$$

is therefore ASD. It satisfies $\mathcal{L}_K g = 0$, where $K = \partial_z$. Using the relations

$$*_h dt = dt \wedge dy, \quad *_h dy = 2dt \wedge dx, \quad *_h dx = dy \wedge dx + 2H_x dy \wedge dt$$

we verify that equation (5.28) is equivalent to $d *_h (d + \nu/2)(W_x) = 0$. Therefore

$$W_{xx} dy \wedge dx + (2(H_x W_x)_x - W_{tx}) dy \wedge dt + 2W_{xy} dt \wedge dx = d\alpha,$$

and we deduce that $V = W_x$ is a solution to the monopole equation (5.23) on the EW background given by (5.25). We choose a gauge in which $\alpha = Q dy + P dt$. This yields

$$Q_x = -W_{xx}, \quad P_x = -2W_{xy}, \quad P_y - Q_t = 2(H_x W_x)_x - W_{xt}, \quad (5.31)$$

so $Q = -W_x + A(y, t)$, $P = -2W_y + B(y, t)$ and $\alpha = -W_x dy - 2W_y dt + A dy + B dt$. The integrability conditions $P_{xy} = P_{yx}$ are given by (5.28), and $A_t = B_y$. Therefore there exists $C(y, t)$ such that $A = C_y, B = C_t$. We now replace z by $z - C$ and the metric (5.30) becomes (5.29). This proves that (5.29) is ASD. It is also scalar-flat, because, as a consequence of (5.28),

$$R = 8(W_{xyy} - W_{xxt} + (H_x W_x)_{xx})W_x = 0. \quad (5.32)$$

We now choose the null tetrad

$$\begin{aligned} e^{00'} &= -2W_x dt, \quad e^{10'} = \frac{dz - 2W_y dt}{2W_x}, \\ e^{01'} &= dz - 2W_x dy - 2W_y dx + ze^{00'}, \quad e^{11'} = dx + H_x dy + ze^{10'}, \end{aligned}$$

³ With definition $u = H_x$ the x derivative of equation (5.27) becomes the dispersionless Kadomtsev–Petviashvili equation (5.26) originally used in [5]. There are some computational advantages in working with the ‘potential’ form (5.27).

such that $g = 2(e^{00'}e^{11'} - e^{10'}e^{01'})$. The basis of SD two forms $\Sigma^{A'B'}$ is given by

$$\begin{aligned}\Sigma^{0'0'} &= \iota_{A'}\iota_{B'}\Sigma^{A'B'} = e^{00'} \wedge e^{10'} = dz \wedge dt \\ \Sigma^{0'1'} &= \iota_{A'}o_{B'}\Sigma^{A'B'} = e^{10'} \wedge e^{01'} - e^{00'} \wedge e^{11'} = dt \wedge d(z^2) + 2dt \wedge dW + dy \wedge dz. \\ \Sigma^{1'1'} &= o_{A'}o_{B'}\Sigma^{A'B'} = e^{01'} \wedge e^{11'} = 2W_x dx \wedge dy + 2(zW_x + W_y)dx \wedge dt - dx \wedge dz \\ &\quad + (2H_x W_x - 2zW_y)dt \wedge dy + z dz \wedge dy + (H_x + z^2)dz \wedge dt.\end{aligned}$$

These two-forms satisfy

$$\begin{aligned}-2\Sigma^{0'0'} \wedge \Sigma^{1'1'} &= \Sigma^{0'1'} \wedge \Sigma^{0'1'}, \quad d\Sigma^{0'0'} = 0, \quad d\Sigma^{0'1'} = 0, \\ d\Sigma^{1'1'} &= d(H_x - 2W) \wedge dt \wedge dz + (W_{xt} - W_{yy} - (H_x W_x)_x)dx \wedge dy \wedge dt.\end{aligned}\tag{5.33}$$

Therefore Proposition 2.3 implies that the metric (5.29) admits a constant spinor which is preserved by $K = \partial_z$.

Converse : Let g be a real analytic ASD metric with a covariantly constant spinor $\iota_{A'}$, which is Lie derived along a Killing vector K . Theorem 4.3 implies that the corresponding twistor space \mathcal{PT} is equipped with $l \in \Gamma(\kappa^{-1/4})$. The Killing vector K gives rise to a holomorphic vector field on \mathcal{PT} which preserves l . Therefore the minitwistor space \mathcal{Z} (the space of trajectories of K in \mathcal{PT}) also admits a preferred real section of the $-1/4$ power of its canonical bundle. The minitwistor space \mathcal{Z} satisfies the assumptions of Theorem 5.2 and the corresponding EW metric is of the form $\hat{g} = \Omega^2 g$, where g is given by (5.29). Both \hat{g} and g are scalar flat (this follows from the spinor Ricci identities and from equation (5.32) respectively). As a consequence we deduce that $\Omega = \Omega(t)$. Now we can use the coordinate freedom [5] to absorb Ω in the solution to the equation (5.27). □

Corollary 5.4 *Let (\mathcal{M}, g) be an ASD null-Kähler manifold with symmetry. Then the Einstein–Weyl structure induced by (5.22) on the space of orbits of this symmetry is locally of the form (5.25).*

Remarks:

- Theorem 5.3 is analogous to a result of LeBrun [12] who constructs all scalar-flat Kähler metrics with symmetry in Euclidean signature from solutions to the $SU(\infty)$ Toda equation and its linearisation.
- A class of solutions to the monopole (5.28) can be obtained from vectors tangent to a space of solutions to (5.27) generated by

$$x \rightarrow a(\epsilon)x + b(\epsilon), \quad t \rightarrow a(\epsilon)t + c(\epsilon), \quad y \rightarrow a(\epsilon)y + e(\epsilon),$$

where $a(0) = a, b(0) = b, c(0) = c, e(0) = e$. The corresponding linearised solution is given by

$$W(x, y, t) = \frac{dH}{d\epsilon}|_{\epsilon=0} = a(xH_x + yH_y + tH_t) + bH_x + cH_t + eH_y.$$

- If $H = \text{const}$ then (5.28) reduces to the wave equation in $2 + 1$ dimension, and consequently the metric (5.29) is the $(++--)$ Gibbons–Hawking solution [9].

- Note that $d\Sigma^{1'1'} \neq 0$ unless $W = H_x/2 + f(t)$, in which case

$$d\Sigma^{1'1'} = d(H_{xt} - H_x H_{xx} - H_{yy}) \wedge dy \wedge dt = 0,$$

and we are working in a covariantly constant real spin frame. The metric

$$g = \frac{H_{xx}}{2}(dy^2 - 4dxdt - 4H_x dt^2) - \frac{2}{H_{xx}}(dz - \frac{H_{xx}dy}{2} - H_{xy}dt)^2 \quad (5.34)$$

is therefore pseudo hyper-Kähler. In [5] it was shown that all pseudo hyper-Kähler metrics with a symmetry satisfying $dK_+ \wedge dK_+ = 0$ are locally given by (5.34). Here dK_+ is a self-dual part of dK .

- If $W_x \neq H_{xx}/2$ then (5.29) is not Ricci-flat. This can be verified by a direct calculation. It also follows from more geometric reasoning: The Killing vector $K = \partial_z$ acts on SD two-forms by a Lie derivative. One can choose a basis $\Sigma^{A'B'}$ such that one element of this basis is fixed, and the Killing vector rotates the other two. The components of the SD derivative of K are coefficients of these rotations. Therefore $(dK)_+ = \text{const}$ if g is pseudo hyper-Kähler. In our case $dK_+ = (H_{xx}/W_x)dz \wedge dt$. Therefore H_{xx}/W_x must be constant for (5.29) to be Ricci-flat. An example of a non-vacuum metric is given by $W = H_y/2$.

5.3 Pseudo hyper-Kähler metrics with symmetry

In this subsection I shall assume that an ASD null-Kähler structure (\mathcal{M}, g) admits an additional parallel spinor $o_{A'}$ such that $o_{A'}\iota^{A'} = 1$. Now there exists a covariantly constant basis of the spin space $S^{A'}$, and (\mathcal{M}, g) is pseudo hyper-Kähler. In the split signature we can arrange for one of the complex structures to be real and for the other two to be purely imaginary:

$$-I^2 = S^2 = T^2 = 1, \quad IST = 1,$$

and S and T determine a pair of transverse null foliations. Now

$$g(X, Y) = g(IX, IY) = -g(SX, SY) = -g(TX, TY)$$

for any pair of real vectors X, Y . The endomorphism I endows \mathcal{M} with the structure of a two-dimensional complex Kähler manifold, as does every other complex structure $aI + bS + cT$ parametrised by the points of the hyperboloid $a^2 - b^2 - c^2 = 1$. Using the identification between the two-forms, and endomorphisms given by g we can write

$$S = \Sigma^{0'0'} - \Sigma^{1'1'}, \quad I = \Sigma^{0'0'} + \Sigma^{1'1'}, \quad T = \Sigma^{0'1'}.$$

Killing vectors on pseudo hyper-Kähler spaces give rise to a homomorphism

$$S^1 \rightarrow SO(2, 1).$$

Therefore K induces an S^1 action on the hyperboloid. Assume that $g(K, K) \neq 0$. If the action is trivial, then we are dealing with the $(++--)$ Gibbons-Hawking ansatz [9]. Otherwise, there always exists a fixed point of the S^1 action. If the two-form corresponding to this fixed point is non-degenerate, then g is given in terms of solutions to the $SU(\infty)$ equation [8, 12, 21]. Finally if the fixed point corresponds to a degenerate two-form, then g is given by (5.34). Pseudo hyper-Kähler metrics which admit K such that $g(K, K) = 0$, and $\mathcal{L}_K g = \mathcal{L}_K I = \mathcal{L}_K S = \mathcal{L}_K T = 0$ have been found in [1]. Conformal symmetries of pseudo hyper-Kähler metrics with non-null self-dual derivative have been classified in [6]

6 Acknowledgements

I am grateful to David Calderbank, Nigel Hitchin, Lionel Mason, Maciej Przanowski, David Robinson and Paul Tod for useful discussions. I would like to thank *Centro de Investigacion y de Estudios Avanzados* in Mexico, where part of this work was done, for the hospitality. My visit to Mexico was partially supported by grants 32427-E Proyecto de CONACYT and LMS 5619.

7 Appendix—spinor notation

Let \mathcal{M} be a real four-manifold equipped with a $(++--)$ metric g and compatible volume form ν . We use the conventions of Penrose and Rindler [16]. a, b, \dots are four-dimensional vector indices and $A, B, \dots, A', B', \dots$ are two-dimensional spinor indices. They have ranges $0, 1$ and $0', 1'$ respectively. The tangent space at each point of \mathcal{M} is isomorphic to a tensor product of the two real spin spaces

$$T\mathcal{M} = S^A \otimes S^{A'}. \quad (7.35)$$

This isomorphism is given by

$$V^a \longrightarrow V^{AA'} = \begin{pmatrix} V^0 + V^3 & V^1 + V^2 \\ V^1 - V^2 & V^0 - V^3 \end{pmatrix}.$$

Orthogonal transformations decompose into products of ASD and SD rotations

$$SO(2, 2) = (SL(2, \mathbb{R}) \times \widetilde{SL}(2, \mathbb{R}))/\mathbb{Z}_2. \quad (7.36)$$

The Lorentz transformation $V^a \longrightarrow \Lambda^a_b V^b$ is equivalent to

$$V^{AA'} \longrightarrow \lambda^A_B V^{BB'} \lambda^{A'}_{B'},$$

where λ^A_B and $\lambda^{A'}_{B'}$ are elements of $SL(2, \mathbb{R})$ and $\widetilde{SL}(2, \mathbb{R})$.

Spin dyads (o^A, ι^A) and $(o^{A'}, \iota^{A'})$ span S^A and $S^{A'}$ respectively. The spin spaces S^A and $S^{A'}$ are equipped with parallel symplectic forms ε_{AB} and $\varepsilon_{A'B'}$ such that $\varepsilon_{01} = \varepsilon_{0'1'} = 1$. These anti-symmetric objects are used to raise and lower the spinor indices via $\iota_A = \iota^B \varepsilon_{BA}$, $\iota^A = \varepsilon^{AB} \iota_B$. We shall use normalised spin frames so that

$$o^B \iota^C - \iota^B o^C = \varepsilon^{BC}, \quad o^{B'} \iota^{C'} - \iota^{B'} o^{C'} = \varepsilon^{B'C'}.$$

Let $e^{AA'}$ be a null tetrad of one-forms on \mathcal{M} i.e.

$$g = \varepsilon_{AB} \varepsilon_{A'B'} e^{AA'} e^{BB'} = 2(e^{00'} e^{11'} - e^{10'} e^{01'}),$$

and let $\nabla_{AA'}$ be the frame of vector fields dual to $e^{AA'}$. The orientation is given by fixing the volume form

$$\nu = e^{01'} \wedge e^{10'} \wedge e^{11'} \wedge e^{00'}.$$

Apart from orientability \mathcal{M} must satisfy some other topological restrictions for $++--$ metric, and a global spinor fields to exist. We shall not take them into account as we work locally in \mathcal{M} .

Any two-form Ω_{ab} can be written as

$$\Omega_{ABA'B'} = \Omega_{AB}\varepsilon_{A'B'} + \tilde{\Omega}_{A'B'}\varepsilon_{AB}$$

where Ω_{AB} are $\tilde{\Omega}_{A'B'}$ symmetric in their indices since Ω_{ab} is skew. This is the decomposition of a two-form into its ASD and SD part. The space of SD two forms is therefore isomorphic to a symmetric tensor product of two primed spin spaces.

The local basis Σ^{AB} and $\Sigma^{A'B'}$ of spaces of ASD and SD two-forms are defined by

$$e^{AA'} \wedge e^{BB'} = \varepsilon^{AB}\Sigma^{A'B'} + \varepsilon^{A'B'}\Sigma^{AB}. \quad (7.37)$$

The first Cartan structure equations are

$$de^{AA'} = e^{BA'} \wedge \Gamma^A_B + e^{AB'} \wedge \Gamma^{A'}_{B'},$$

where Γ_{AB} and $\Gamma_{A'B'}$ are the $SL(2, \mathbb{R})$ and $\widetilde{SL}(2, \mathbb{R})$ spin connection one-forms. They are symmetric in their indices, and

$$\Gamma_{AB} = \Gamma_{CC'AB}e^{CC'}, \quad \Gamma_{A'B'} = \Gamma_{CC'A'B'}e^{CC'}, \quad \Gamma_{CC'A'B'} = o_{A'}\nabla_{CC'}\iota_{B'} - \iota_{A'}\nabla_{CC'}o_{B'}.$$

The curvature of the spin connection

$$R^A_B = d\Gamma^A_B + \Gamma^A_C \wedge \Gamma^C_B$$

decomposes as

$$R^A_B = C^A_{BCD}\Sigma^{CD} + (1/12)R\Sigma^A_B + \Phi^A_{BC'D'}\Sigma^{C'D'},$$

and similarly for $R^{A'}_{B'}$. Here R is the Ricci scalar, $\Phi_{ABA'B'}$ is the trace-free part of the Ricci tensor R_{ab} ,

$$-2\Phi_{ABA'B'} = R_{ab} - \frac{1}{4}Rg_{ab},$$

and C_{ABCD} is the ASD part of the Weyl tensor

$$C_{abcd} = \varepsilon_{A'B'}\varepsilon_{C'D'}C_{ABCD} + \varepsilon_{AB}\varepsilon_{CD}C_{A'B'C'D'}.$$

A conformal structure is called ASD iff $C_{A'B'C'D'} = 0$. Define the operators Δ_{AB} and $\Delta_{A'B'}$ by

$$[\nabla_a, \nabla_b] = \varepsilon_{AB}\Delta_{A'B'} + \varepsilon_{A'B'}\Delta_{AB}.$$

The spinor Ricci identities are

$$\Delta_{AB}\iota_{A'} = \Phi_{ABA'B'}\iota^{B'} \quad (7.38)$$

$$\Delta_{A'B'}\iota_{C'} = [C_{A'B'C'D'} - \frac{1}{12}R\varepsilon_{D'(A'}\varepsilon_{B')C'}]\iota^{D'}, \quad (7.39)$$

(and analogous equations for unprimed spinors). Bianchi identities translate to

$$\nabla^{A'}_A C_{A'B'C'D'} = \nabla^B_{(B'}\Phi_{C'D')AB}, \quad \nabla^{AA'}\Phi_{ABA'B'} + \frac{1}{8}\nabla_{BB'}R = 0. \quad (7.40)$$

Let K be a pure Killing vector. Then $\nabla^{(A'}_A K^{B')}_B = 0$, $\nabla^a K_a = 0$. This implies

$$\nabla_a K_b = \phi_{AB}\varepsilon_{A'B'} + \psi_{A'B'}\varepsilon_{AB},$$

where $\psi_{A'B'}$ and ϕ_{AB} are symmetric spinors. The well known identity $\nabla_a \nabla_b K_c = R_{bcad} K^d$ implies

$$\nabla^A{}_{A'} \psi_{B'C'} = -2C_{A'B'C'}^{D'} K_{D'}^A - 2K^B{}_{(A'} \Phi_{B'C')}^A + \frac{R}{6} \varepsilon_{A'(B'} K^A{}_{C')} - \frac{4}{3} \varepsilon_{A'(B'} \Phi_{C')}^{D'DA} K_{DD'}. \quad (7.41)$$

Therefore in an ASD vacuum $\psi_{A'B'} = \text{const.}$ A Lie derivative of a spinor along a Killing vector K is given by

$$\mathcal{L}_K \iota^{A'} = K^b \nabla_b \iota^{A'} - \psi_{B'}^{A'} \iota^{B'}.$$

Note that if $\iota_{A'}$ is covariantly constant, and the Killing vector K preserves the $\iota_{A'}$ we deduce that $\psi_{A'B'} = \psi \iota_{A'} \iota_{B'}$ for some function ψ . The identity (7.41) yields

$$\nabla_{AA'} \psi = \frac{1}{2} F_{AB} K_{A'}^B$$

so $\psi \neq \text{const}$ unless g is hyper-Kähler.

References

- [1] Barrett, J. Gibbons, G.W., Perry, M.J., Pope, C.N., & Ruback, P.J. (1994) Kleinian Geometry and the N=2 Superstring, *Int.J.Mod.Phys.* **A9**, 1457-1494.
- [2] Bryant, R. L. (2000) Pseudo-Riemannian metrics with parallel spinor fields and vanishing Ricci tensor. Global analysis and harmonic analysis, 53–94, *Semin. Congr.*, **4**, Soc. Math. France, Paris.
- [3] Dancer, A. (2000) Hyper-Kähler manifolds, in *Surveys in Differential Geometry, Vol VI Essays on Einstein Manifolds* Edited by: Claude LeBrun, and McKenzie Wang.
- [4] Dunajski, M. (2000) ASD Null Kähler metrics with symmetry, *Twistor Newsletter* **45**, 62-65.
- [5] Dunajski, M., Mason, L.J., & Tod, K.P. (2001) Einstein–Weyl geometry, the dKP equation and twistor theory *J. Geom. Phys.* **37**, 63-92.
- [6] Dunajski, M. & Tod K.P. (2001) Einstein–Weyl Structures from Hyper–Kähler Metrics with conformal Killing Vectors, *Diff. Geom. Appl.* **14**, 39-55.
- [7] Garcia, A., Plebański, J. F.& Robinson, I. (1977) Null strings and complex Einstein-Maxwell fields with cosmological constant, *Gen. Rel. Grav.* **8**, 841-854.
- [8] Finley, J.D. & Plebański, J.F. (1979) The classification of all \mathcal{H} spaces admitting a Killing vector, *J. Math. Phys.* **20**, 1938.
- [9] Gibbons, G. W.& Hawking S.W. (1978) Gravitational multi-instantons, *Phys. Lett.* **78B**, 430-432.
- [10] Jones, P. & Tod, K.P. (1985) Minitwistor spaces and Einstein-Weyl spaces, *Class. Quantum Grav.* **2** 565-577.
- [11] Kramer, D., Stephani, H., MacCallum, M., & Herlt, E. (1980) *Exact solutions of Einstein's field equations*, CUP.
- [12] LeBrun, C.R. (1991) Explicit self-dual metrics on $\mathbb{CP}^2 \# \dots \# \mathbb{CP}^2$, *J. Diff. Geom.* **34** 233-253.

- [13] Mason, L.J. & Woodhouse, N.M.J. (1996) *Integrability, Self-Duality, and Twistor Theory*, L.M.S. Monographs New Series, **15**, OUP.
- [14] Ooguri, H. & Vafa, C. (1990) Self-Duality and N=2 String Magic Int. J. Mod. Phys. **A5** 1389-1398.
- [15] Penrose, R. (1976) Nonlinear gravitons and curved twistor theory, Gen. Rel. Grav. **7**, 31-52.
- [16] Penrose, R. & Rindler (1986) *Spinors and Space-Time*, Vol **1, 2**, CUP.
- [17] Plebański, J. F. (1975) Some solutions of complex Einstein Equations, J. Math. Phys. **16** 2395-2402.
- [18] Pontecorvo, M. (1992) On twistor spaces of anti-self-dual Hermitian surfaces, Trans. Am. Math. Soc. **331**, 653-661.
- [19] Robinson, D.C (2000) Holomorphic 4-metrics and real structures, preprint KCL-MTH-OO-59
- [20] Ward, R.S. (1985) Integrable and solvable systems and relations among them, Phil. Trans. R. Soc. A **315**, 451-457.
- [21] Ward, R.S. (1990) Einstein–Weyl spaces and $SU(\infty)$ Toda fields, Class. Quantum Grav. **7** L95-L98.
- [22] Woodhouse, N.M.J. (1992) Contour integrals for the ultrahyperbolic wave equation. Proc. Roy. Soc. London Ser. **A 438** no. 1902, 197–206.